Design and Analysis of Algorithm
Basics of Complexity Theory

1. Decision Problem

2. Deterministic Computation

3. Several Important Complexity Classes
   - $\mathcal{P}$ vs. $\mathcal{NP}$
   - $\mathcal{NP}$-complete

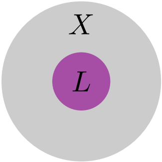
4. Randomized Computation
   - $\mathcal{BPP}$

Outline

Decision Problem

Decision Problem: recognition of a set of strings $L \subseteq X$

- X: a set of strings
- x: a string in X (each string corresponds to an instance)
- L: language (a subset of X satisfying some property)



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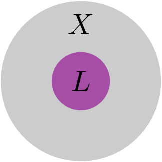

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Example

- $X = \mathbb{N}$
- L are Primes $= \{2, 3, 5, 7, 11, 13, \dots\}$
- decide if x is a prime.

Motivation for Complexity Theory

We always want to know if a given problem can be *efficiently* solved by an algorithm.

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1. Precisely model algorithms
   - What is computation?
   - What is computable?
2. Precisely define what does it means for efficient.

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Turing Machine

1936, London Mathematical Society: On computable numbers, with an application to the Entscheidungsproblem.

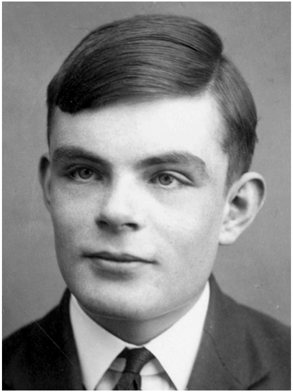
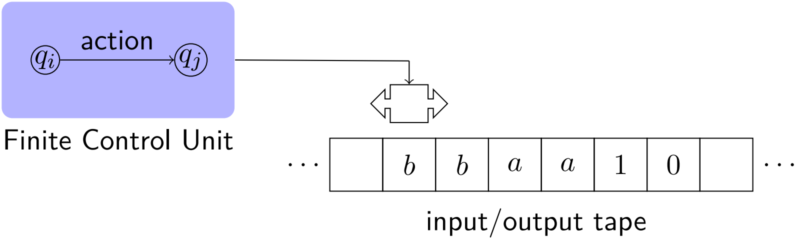


Figure: Alan Turing

Turing Machine

Turing machine: automatic machine that has a tape (divided into infinite cells), a control unit and a read/write head.



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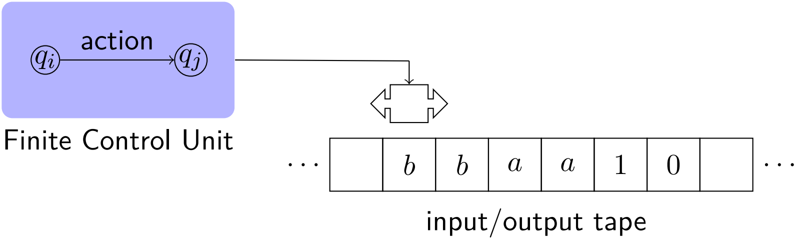

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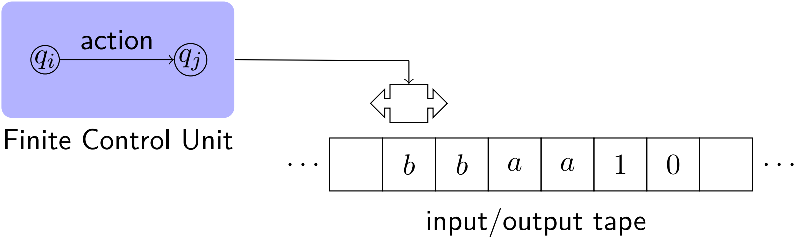
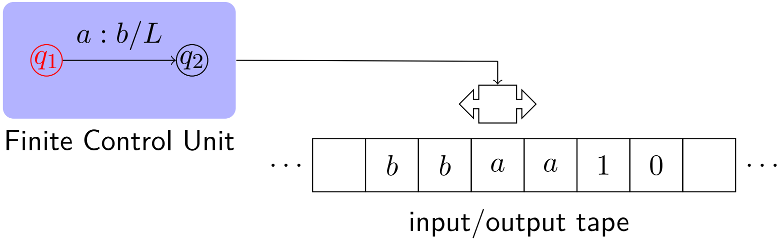


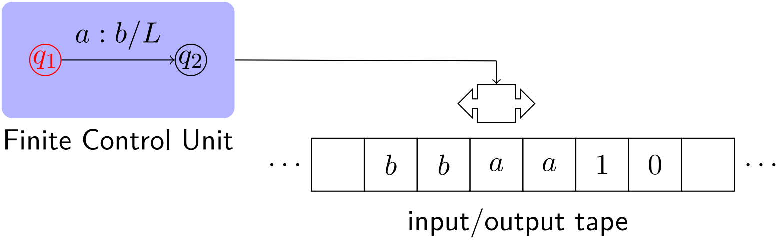
- At the beginning, the tape contains the input in several cells. Other places are empty.
- During computation, the control unit monitor current state and the head value, can do the following operations:
  1. wipe off old value and write new values
  2. change the current state
  3. move head left or right

An Example



$a : b/L$
q_1
q_2
Finite Control Unit
$\cdots$
b
b
a
a
1
0
$\cdots$
input/output tape

An Example



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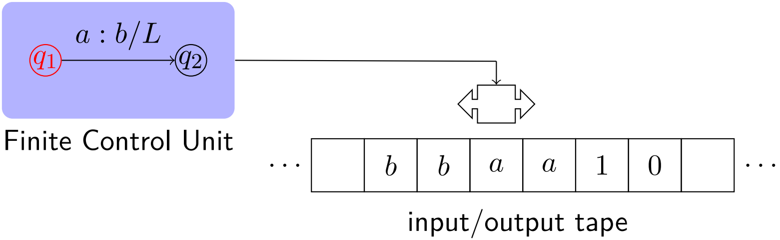

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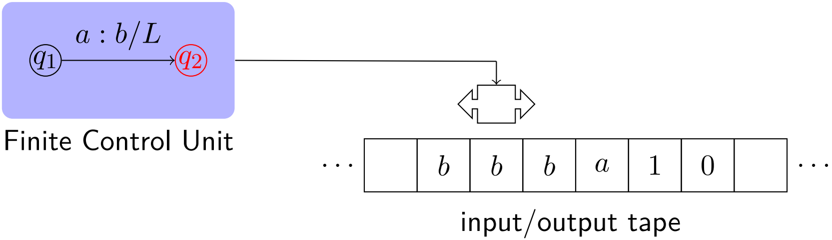

Intuition of Turing Machine

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Mimic how human being solve a problem

- TM has a finite number of states (memory)
- TM is provided a tape, which contains infinite cells (paper)
- a symbol can be scanned from a cell or printed to a cell (reading and writing)

Formal Definition

Definition 1 (Turing Machine)

TM consists $(Q, \Sigma, \Gamma, \delta, q_0, q_{\mathsf{acc}}, q_{\mathsf{rej}})$

- Q: a finite set of states
- Σ: input alphabets
- Γ: working alphabets (including $\perp$, $\Sigma \subseteq \Gamma$)
- q_0: the initial state of Q;
- $q_{\mathsf{acc}}, q_{\mathsf{rej}}$: accept and reject state of Q
- δ: transition function

$$\delta : (Q \backslash \{q_{\mathsf{acc}}, q_{\mathsf{rej}}\}) \times \Gamma \to Q \times \Gamma \times \{L, R\}$$

Running Time of TM

Definition 2

We denote the running time of TM by $t_M(n)$, which is the maximum steps that TM runs on all inputs of length n

Polynomial Time

$$\bigcup_{k \in \mathbb{N}} \mathsf{TIME}(n^k)$$

The Extended Church-Turing Thesis

 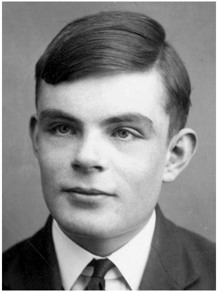

Figure: Alonzo Church & Alan Turing

Everyone's intuition of **Efficient** Algorithms = **Polynomial-Time** deterministic TMs

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Non-determinism doesn't give TM any power to recognize more languages.

- Any NDTM can be simulated by a TM (with potentially exponential time overhead) by trying all branches of the NDTM machine "in parallel" by using BFS.

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Why TMs are so powerful?

- TM has a working tape (好记性不如烂笔头)
- TM itself can be treated as data! TM can take another TM as its input.

Universal TM

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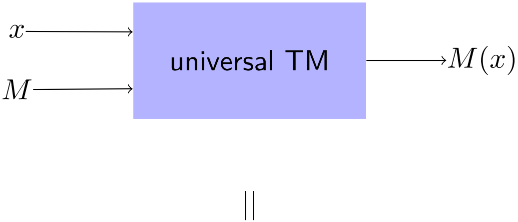

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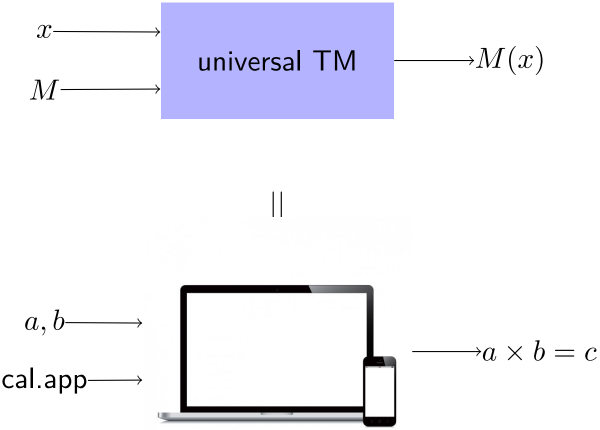

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Next, we introduce two important sets of problems, characterized by time complexity by DTM and NDTM:

$\mathcal{P}$ and $\mathcal{NP}$

$\mathcal{P}$ Complexity

Definition 3 ($\mathcal{P}$ Language)

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Example of $\mathcal{P}$ Languages

- $L = \{\text{even integers}\}$, M just need to check if the last bit is 0.
- $L = \text{PRIME}$, M is the AKS primality test algorithm.

Definition 4 ($\mathcal{NP}$ Languages - Conventional)

$L \in \mathcal{NP}$ if there exists a non-deterministic poly-time TM M:

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Alert

$\mathcal{NP}$ means non-deterministic poly-time, not non-poly-time!

Modern Definition

Definition 5 ($\mathcal{NP}$ Complexity - Modern)

$L \in \mathcal{NP}$ if there exists a deterministic poly-time TM M:

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Equivalence between traditional and modern definitions

- Even though M is a deterministic machine, its second argument w captures the nondeterminism in the definition.

Examples of $\mathcal{NP}$ Language - Composites

$L =$ COMPOSITE

- instance x is an integer
- a witness w for $x \in L$ is a non-trivial factor of x
- M just need to check if w divides x, which could be done in polynomial time.

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In fact, COMPOSITE also belong to $\mathcal{P}$ (think why?)

Examples of $\mathcal{NP}$ Language - SAT and 3-SAT

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Example of 3-SAT

- instance $\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$
- witness: $x_1 = 1$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$

Examples of $\mathcal{NP}$ Language - Hamilton Path

Hamilton Graph: Given an undirected graph $G = (V, E)$, does there exists a simple path that visits every node?

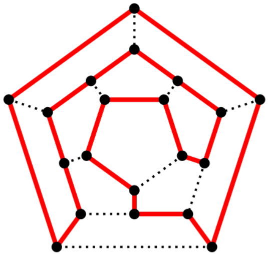

Figure: Hamiltonian Graph (a path traverses through each verticals exactly once)

witness: a path

M check if the path contains each node in V exactly once

$\mathcal{P}$ vs. $\mathcal{NP}$

As per definition, $\mathcal{P} \subseteq \mathcal{NP}$. Because $L \in \mathcal{P} \Rightarrow L \in \mathcal{NP}$:

- $M'(x, w)$ can always sets $w = \perp$ and decide whether $x \in L$ using M.
- Alternatively, "short" M can be viewed as a witness for $x \in L$. Think about why the description of M is short?

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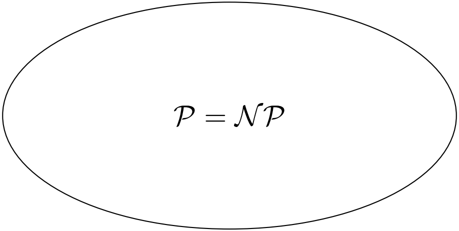
1971: Cook, Edmonds, Levin, Yablonski, Gödel

Perhaps the most prominent question in TCS:

$$\mathcal{P} =? \mathcal{NP}$$

$\mathcal{P} = \mathcal{NP}$



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In principle, every aspect of life could be efficiently and globally optimized $\cdots$

$\cdots$ life as we know it would be different!

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Define a collection of languages $L_i = \{(y, z) | \exists w \text{ s.t. } y = f(z||w)\}$, where $z \in \{0,1\}^i$, $w \in \{0,1\}^{n-i}$

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Algorithm 4: Invert(y)

```
z = ε;
for i ← 1 to n do
    if (y, z||0) ∈ L_i then z = z||0;
    else z = z||1;
end
return z
```

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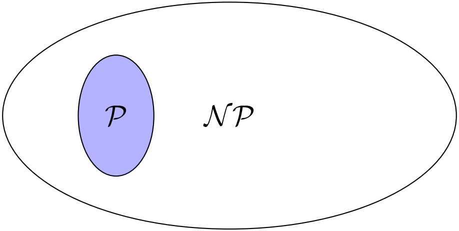
Warning

OWFs do not exist *does not imply* $\mathcal{P} = \mathcal{NP}$

Consensus



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Conjecture: $\underbrace{\text{No poly-time algorithm}}_{\text{intractable}}$ for 3-SAT

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Motivation of Reduction

$\mathcal{NP}$ is the set of many problems.

How to figure out the relations among them?

A central approach is finding reductions

Polynomial Time Reducibility

Language L' is *poly-time reducible* or *reduces* to language L, written as $L' \leq_p L$, if there is a determinstic poly-time function $\mathcal{R} : L' \rightarrow L$ so that:

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We should pay attention to:

- the direction of $\mathcal{R}$
- the time complexity of $\mathcal{R}$

$\mathcal{NP}$-Hard

Definition 6 ($\mathcal{NP}$-Hard)

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Fact: languages in $\mathcal{NP}$-hard may not fall in $\mathcal{NP}$.

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Definition Intuition: $\mathcal{NP}$-complete represents the set of hardest problems in $\mathcal{NP}$.

- We can solve all problems in $\mathcal{NP}$ if we find an efficient algorithm for any problems in $\mathcal{NP}$-complete.

Theorem 8

Suppose $Y \in \mathcal{NP}$*-complete, then* $Y \in \mathcal{P} \iff \mathcal{P} = \mathcal{NP}$.

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- This theorem essentially states that if $\mathcal{P} \cap \mathcal{NPC}$ is non-empty iff $\mathcal{P} = \mathcal{NP}$.

$\mathcal{P}$ vs. $\mathcal{NP}$ revisited

Overwhelming consensus (still): $\mathcal{P} \neq \mathcal{NP}$.

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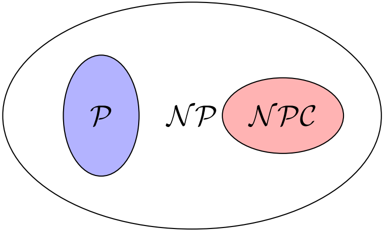


Figure: $\mathcal{P} \neq \mathcal{NP}$

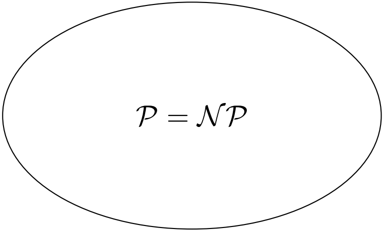
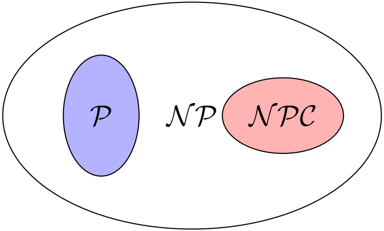


Figure: $\mathcal{P} = \mathcal{NP}$

$\mathcal{P}$ vs. $\mathcal{NP}$ revisited

Overwhelming consensus (still): $\mathcal{P} \neq \mathcal{NP}$.



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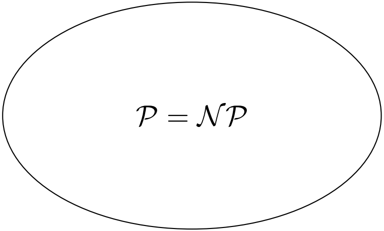
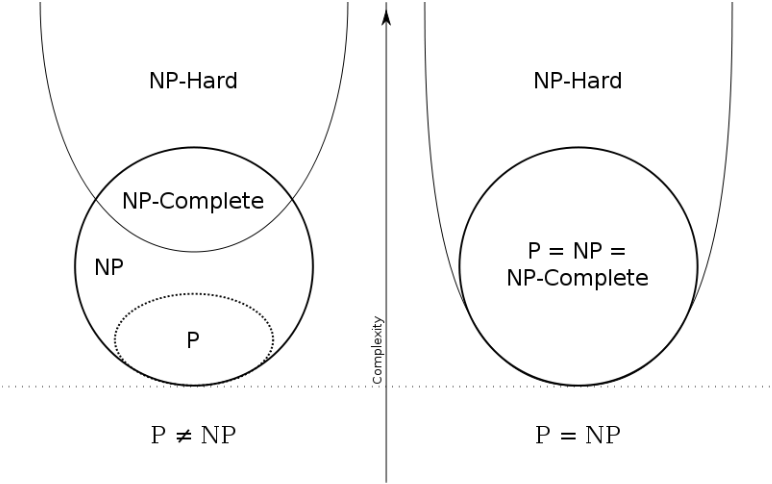


Figure: $\mathcal{P} = \mathcal{NP}$

Why we believe $\mathcal{P} \neq \mathcal{NP}$? Because some problems appear significantly harder.


NP-Hard
NP-Complete
NP
P
Complexity
NP-Hard
P = NP =
NP-Complete
P ≠ NP
P = NP

Outline

Motivation of Randomized Algorithm

TM models deterministic algorithms.

TM does not seem to capture one aspect of reality — the ability to make random choices during computation

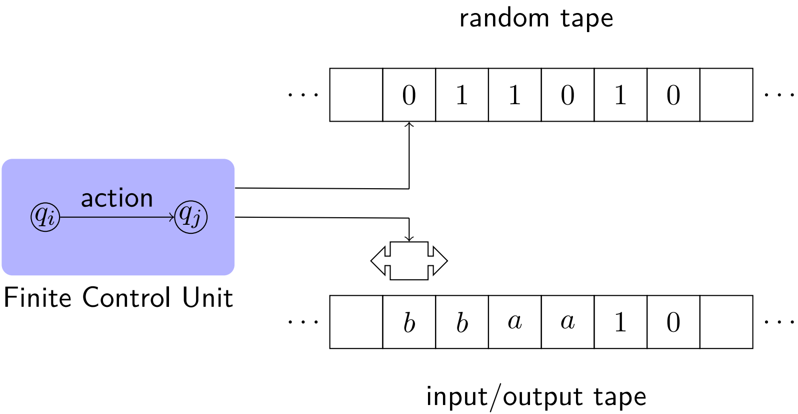
- Most programming languages provide a built-in RNG.

It makes sense to consider algorithms that can toss a coin, a.k.a. use a source of random bits. Such algorithms have been implicitly studied for a long time.

- estimate facts about a large sample by taking a small sample
- simulate real-world systems that are themselves probabilistic, such as nuclear fission and the stock market
- differential equations

Probabilistic Turing Machine

Probabilistic Polynomial-time TM models probabilistic algorithm.



PTM vs. NDTM

NDTM is a TM with two transition functions. PTM is syntactically similar.

The difference is in how we interpret the working of TM.

- In a PTM, each transition is taken with probability $1/2$, a computation that runs for time t gives rise 2^t branches in the graph of all computations, each of which is taken with probability $1/2^t$. $\Pr[M(x) = 1]$ is simply the *fraction* of branches that end with M outputting a 1.
- In a NDTM, $M(x) = 1$ iff there exists a branch that outputs 1

On a conceptual level, PTM and NDTM are very different

- PTM like TM and unlike NDTM, is intended to model realistic computation devices.

Outline

Bounded-Error Probabilistic Polynomial Time

Definition 9 ($\mathcal{BPP}$ Complexity)

$L \in \mathcal{BPP}$ iff there exists a probabilistic polynomial time TM M such that:

$$\forall x \in L : \quad \Pr[M(x) = 1] \geq \alpha$$
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Bounded-error Probabilistic Polynomial Time (weak version)

- A typical choices is $\alpha = 2/3$, $\beta = 1/3$. In this case, the class of decision problems solvable by a probabilistic TM in polynomial time with an error probability e bounded away from $1/3$ for all instances

Reduce the Error (1/2)

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- It does not even have to be constant: e could be as high as $1/2 - n^{-c}$ on one hand, or as small as 2^{-n^c} on the other hand, where c is any positive constant, and n is the length of input.
- The idea is if the algorithm is run many times, the chance that the majority of the runs are wrong drops off exponentially as a consequence of the Chernoff bound.

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Chernoff Bounds (Lower Tail): Let $X = \sum_{i=1}^{n} X_i$, $\Pr[X_i] = p$, $\mu = \mathbb{E}(X) = np$.

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Do the Majority Vote, i.e., set $(1-\delta)\mu = n/2$ and thus $\delta = 1 - 1/2p$, we obtain:

$$\Pr[X \leq n/2] \leq e^{-n\frac{(1-2p)^2}{8p}}$$

About $\mathcal{BPP}$

$\mathcal{BPP}$ is one of the largest practical class of problems, since problems in $\mathcal{BPP}$ have efficient probabilistic algorithms that can be run quickly on real modern machines.

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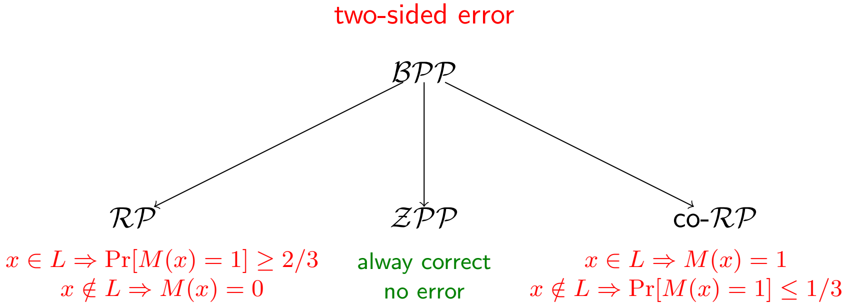
[Agrawal, Kayal, Saxena 2002]: gave a deterministic polynomial-time algorithm for PRIME, thus showing that it is in $\mathcal{P}$.

One-sided and Zero-sided Error

$\mathcal{ZPP}$: probabilistic polynomial-time TM always returns correct YES or NO answer, or halts with low probability, a.k.a. running time is polynomial in expectation for every input

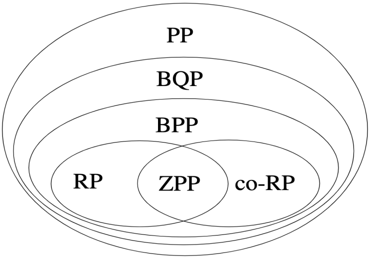


- $\mathcal{BPP}$: Monte Carlo algorithms (probabilistic) likely to be correct in strict polynomial running time
- $\mathcal{ZPP}$: Las Vegas algorithms (probabilistic) are always correct in expected polynomial running time

$\mathcal{BPP}$ in Relation to Other Probabilistic Complexity Classes

$\mathcal{BQP}$ (bounded-error quantum polynomial time): the class of decision problems solvable by a quantum TM in polynomial time with bounded error

- It is the quantum analogue of $\mathcal{BPP}$



Limits of $\mathcal{BPP}$

Consensus: $\mathcal{P} \subseteq \underline{\mathcal{ZPP} = \mathcal{RP} \cap \text{co-}\mathcal{RP}} \subseteq \mathcal{BPP} \subseteq \mathcal{NP}$

$\mathcal{P} \subseteq \mathcal{BPP}$

- An important example of a problem in $\mathcal{BPP}$ still not known to be in $\mathcal{P}$ is polynomial identity testing — determining whether a polynomial is identically equal to the zero polynomial, when you have access to the value of the polynomial for any given input, but not to the coefficients.

$\mathcal{BPP} \subseteq \mathcal{NP}$

- Adleman's theorem: $\mathcal{BPP} \subseteq P/\text{poly}$ (polynomial-size Boolean circuits)
- Karp-Levin theorem: $\mathcal{NP} \subseteq P/\text{poly} \Rightarrow \text{PH} = \sum_2^P$

Thus, $\mathcal{NP} \subseteq \mathcal{BPP}$ will imply collapse of PH, which is unlikely to be true. In other words, $\nexists$ bounded-error probabilistic algorithms for $\mathcal{NPC}$ problems.